

Anomalous transport from holography: Part II

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ABSTRACT: This is a second study of chiral anomaly induced transport within a holographic model consisting of anomalous $U(1)_V \times U(1)_A$ Maxwell theory in Schwarzschild- AdS_5 space-time. In the first part, chiral magnetic/separation effects (CME/CSE) are considered in presence of a static spatially-inhomogeneous external magnetic field. Gradient corrections to CME/CSE are analytically evaluated up to third order in the derivative expansion. Some of the third order gradient corrections lead to an anomaly-induced negative B^2 -correction to the diffusion constant. We also find non-linear modifications to the chiral magnetic wave (CMW). In the second part, we focus on the experimentally interesting case of the axial chemical potential being induced dynamically by a constant magnetic and time-dependent electric fields. Constitutive relations for the vector/axial currents are computed employing two different approximations: (a) derivative expansion (up to third order) but fully non-linear in the external fields, and (b) weak electric field limit but resumming all orders in the derivative expansion. A non-vanishing non-linear axial current (CSE) is found in the first case. Dependence on magnetic field and frequency of linear transport coefficient functions (TCFs) is explored in the second.

KEYWORDS: AdS-CFT Correspondence, Fluid-Gravity Correspondence, Anomaly

ARXIV EPRINT: [TBA](#)

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1 Introduction and summary

Chiral anomalies emerge and play an important role in relativistic QFTs with massless fermions. The anomaly is reflected in three-point functions of currents associated with global symmetries. When the global $U(1)$ currents are coupled to external electromagnetic fields, the triangle anomaly renders the axial current into non-conserved,

$$\partial_\mu J^\mu = 0, \quad \partial_\mu J_5^\mu = 12\kappa \vec{E} \cdot \vec{B}, \quad (1.1)$$

where J^μ/J_5^μ are vector/axial currents, and κ is an anomaly coefficient. For $SU(N_c)$ gauge theory with a massless Dirac fermion in fundamental representation, $\kappa = eN_c/(24\pi^2)$, and e is an electric charge which below will be set to unit. Here \vec{E}, \vec{B} are external electromagnetic fields.

The focus of this paper is in the vector/axial currents induced by external fields in a charged plasma at finite temperature. Presence of triangle anomalies requires modification of usual constitutive relations for the currents. An example of such modification is the chiral magnetic effect (CME) [1–5]¹, that is the induction of an electric current along the applied magnetic field. CME relies on chiral imbalance, which is usually parameterised by an axial chemical potential. Studies of CME can be found in e.g. [9–14] based on perturbative QCD, in e.g. [15–20] within lattice simulations, and in e.g. [21–33] for strongly coupled regime based on the AdS/CFT correspondence [34–36].

The chiral separation effect (CSE) [37, 38] is another interesting phenomenon induced by the anomalies. It is reflected in separation of chiral charges along external magnetic field at finite density of vector charges. Chiral charges can be also separated along external electric field, when both vector and axial charge densities are nonzero, the so-called chiral electric separation effect (CESE) [39, 40].

¹See also [6–8] for earlier related works.

In heavy ion collisions, experimentally observable effects induced by the anomalies were discussed in [41–45]. We refer the reader to [46–50] and references therein for comprehensive reviews on the subject of anomalous transports.

Without the chiral anomaly ($\kappa = 0$), the most general linear in the external fields and charge density off-shell constitutive relation for the vector current has the following form

$$J^t = \rho, \quad \vec{J} = -\mathcal{D}\vec{\nabla}\rho + \sigma_e\vec{E} + \sigma_m\vec{\nabla} \times \vec{B}, \quad (1.2)$$

where ρ is vector charge density and the diffusion \mathcal{D} , electric/magnetic conductivities $\sigma_{e/m}$ are functionals of space-time derivatives. In momentum space they could be resummed into functions of frequency ω and momentum squared q^2 , which we refer to as transport coefficient functions (TCFs). They were studied in depth in [51] for a holographic charged plasma dual to $U(1)$ Maxwell theory in Schwarzschild- AdS_5 .

In [52] we continued this study focusing on transport properties induced by the chiral anomaly. The holographic model was modified to be anomalous $U(1)_V \times U(1)_A$ Maxwell theory. Under various approximations, off-shell constitutive relations were derived for vector/axial currents. In a weak external field approximation, all-order derivatives in the vector/axial currents were resummed into six momenta-dependent TCFs: the diffusion, the electric/magnetic conductivity, and three anomaly-induced TCFs. The latter generalise the chiral magnetic/separation effects. In presence of constant background external fields, nonlinear transports were also revealed. The chiral magnetic effect, including all-order non-linearity in magnetic field, was proven to be exact when all external fields except for a constant magnetic field are turned off. Nonlinear corrections to the constitutive relations due to electric and axial external fields were computed. The resummation of derivatives in constitutive relations was implemented via the technique of [53–56], which was originally invented to resum all-order linear velocity derivatives in the energy-momentum tensor for a holographic conformal fluid. An important new element of our formalism is that it is not based on current conservation (i.e., off-shell formalism), which makes it essentially different from the on-shell formalism of [57]. The constitutive relations and TCFs can be uniquely determined only from dynamical components of the bulk equations. The constraint component is equivalent to continuity equation of thus derived current.

In the present work we continue the study of anomaly-induced transports within the holographic model of [52]. No axial external fields will be turned on in this work. As in [51, 52] we work in the probe limit so that the currents and energy-momentum tensor decouple. In the dual gravity, the probe limit neglects backreaction of the gauge dynamics on the geometry. The holographic model under study consists of two Maxwell fields in the Schwarzschild- AdS_5 black brane geometry. The chiral anomaly is holographically realised via the gauge Chern-Simons actions for both Maxwell fields (with opposite signs). Such a holographic setup can be realised via a top-down brane construction of $D4/D8/\overline{D8}$ [58].

Before diving into the details presented in the following sections, we summarise our main results. When a static but spatially-varying magnetic field is the only external field that is turned on, the constitutive relations for the vector/axial currents are

$$J^t = \rho, \quad J^i = -\frac{1}{2}\partial_i\rho + \frac{12\kappa\mu_5}{2}B_i - G_i(x=\infty), \quad (1.3)$$

$$J_5^t = \rho_5, \quad J_5^i = -\frac{1}{2}\partial_i \rho_5 + \underline{12\kappa\mu B_i} - H_i(x = \infty), \quad (1.4)$$

where ρ/ρ_5 are vector/axial charge densities, the underlined terms in J^i/J_5^i are the chiral magnetic/separation effects. G_i, H_i contain derivatives of ρ, ρ_5, \vec{B} . It is important to stress that (1.3, 1.4) are exact, without any approximations for ρ, ρ_5, \vec{B} . The nonlinearity of the CME/CSE in external magnetic field \vec{B} is completely accounted for by the chemical potentials μ, μ_5 . The non-derivative part of (1.3) is consistent with the “non-renormalisability” of CME [59, 60]. However, as will be clear from (1.6, 1.7), the derivative corrections introduce new effects, which do modify the original CME. Particularly, the currents along the direction of \vec{B} get affected.

When ρ, ρ_5, \vec{B} vary slowly from point to point, G_i, H_i can be calculated order-by-order within derivative expansion. Let us introduce a scaling parameter λ :

$$\partial_\mu = (\partial_t, \partial_i) \longrightarrow (\lambda\partial_t, \lambda\partial_i). \quad (1.5)$$

Then, derivative counting is by powers of λ . Up to second order in the derivative expansion, G_i, H_i are (throughout this work, the electromagnetic fields are thought of as of first order in derivative counting)

$$G_i(x = \infty) = -\frac{\pi}{8}\partial_t\partial_i\rho + \left(\frac{3}{2}\pi + 3\log 2\right)\kappa\partial_t\rho_5 B_i + 18(1 - 2\log 2)\kappa^2(\rho_5^2 + \rho^2) \\ \times \epsilon^{ijk}\partial_j B_k + 18(2 - 3\log 2)\kappa^2\epsilon^{ijk}(\rho_5\partial_j\rho_5 B_k + \rho\partial_j\rho B_k) + \mathcal{O}(\partial^3), \quad (1.6)$$

$$H_i(x = \infty) = -\frac{\pi}{8}\partial_t\partial_i\rho_5 + \left(\frac{3}{2}\pi + 3\log 2\right)\kappa\partial_t\rho B_i + 36(1 - 2\log 2)\kappa^2\rho\rho_5\epsilon^{ijk}\partial_j B_k \\ + 18(2 - 3\log 2)\kappa^2\epsilon^{ijk}(\rho_5\partial_j\rho B_k + \rho\partial_j\rho_5 B_k) + \mathcal{O}(\partial^3). \quad (1.7)$$

Up to $\mathcal{O}(\partial^2)$, the chemical potentials are

$$\mu = \frac{\rho}{2} + \frac{1}{16}(\pi - 2\log 2)\partial^2\rho - \frac{3}{4}(\pi - 2\log 2)\kappa B_k\partial_k\rho_5 + 18(1 - 2\log 2)\kappa^2\rho B^2 + \mathcal{O}(\partial^3), \\ \mu_5 = \mu(\rho \leftrightarrow \rho_5). \quad (1.8)$$

At third order $\mathcal{O}(\partial^3)$, for G_i, H_i we calculated only terms that are linear in ρ, ρ_5 , see (3.22, 3.23) for a complete listing. Among these third order terms, the diffusion constant \mathcal{D}_0 (i.e., the DC limit of the diffusion function \mathcal{D}) gets a negative correction

$$\mathcal{D}_0 = \frac{1}{2} - 18(2\log 2 - 1)\kappa^2 B^2. \quad (1.9)$$

To the best of our knowledge, this is the first anomaly-induced correction to the diffusion constant and being negative it happens to violate the universal form of [61].

With the third order results for J^μ and J_5^μ , we computed the dispersion relation for a free mode that can propagate in the medium:

$$\omega = [\mp 1 + 36(1 - 2\log 2)\kappa^2 \mathbf{B}^2] 6\kappa \vec{q} \cdot \vec{B} - \left[\frac{1}{2} + 18(1 - 2\log 2)\kappa^2 \mathbf{B}^2\right] iq^2 - \frac{i}{8}q^4 \log 2 + \dots \quad (1.10)$$

The first term in (1.10) represents the chiral magnetic wave (CMW) [59]. Interestingly, we see nonlinear in \mathbf{B} corrections to both the speed of CMW and its decay rate. Note that we also expect emergence in (1.10) of the following terms $(\vec{q} \cdot \vec{\mathbf{B}})^2$, $q^2(\vec{q} \cdot \vec{\mathbf{B}})$, $(\vec{q} \cdot \vec{\mathbf{B}})^3$, $q^2(\vec{q} \cdot \vec{\mathbf{B}})^2$ and $(\vec{q} \cdot \vec{\mathbf{B}})^4$. However, our ability to determine coefficients of these terms is limited by the undertaken approximations.

In the second part of this work, we focus on a special setup which is experimentally accessible in condensed matter systems². CME emerges from a nonzero axial chemical potential μ_5 , which is usually assumed to have some background profile. It is, however, possible to induce ρ_5 (and thus μ_5) dynamically through interplay between the electric and magnetic fields, as clear from the continuity equation (1.1). Specifically, we are ready to consider a constant magnetic field $\vec{\mathbf{B}}$ and a time-dependent but spatially-homogeneous electric field $\vec{E}(t)$. For simplicity the charge densities ρ, ρ_5 will be assumed to be spatially-homogeneous too³. From (1.1), ρ could be set to zero. The constitutive relations for the vector/axial currents are

$$J^t = 0, \quad J^i = E_i + \partial_t E_i + 12\kappa\mu_5 \mathbf{B}_i - 12\kappa\epsilon^{ijk} \mathbb{A}_j(1) E_k + \bar{G}_i(x = \infty), \quad (1.11)$$

$$J_5^t = \rho_5, \quad J_5^i = 12\kappa\mu \mathbf{B}_i - 12\kappa\epsilon^{ijk} \mathbb{V}_j(1) E_k + \bar{H}_i(x = \infty), \quad (1.12)$$

where $\mathbb{V}_j(1)$, $\mathbb{A}_j(1)$, \bar{G}_i and \bar{H}_i depend on ρ_5 , \vec{E} and $\vec{\mathbf{B}}$ nonlinearly and will be computed below. Our study is further split into two parts. In section 4.1, $\mathbb{V}_j(1)$, $\mathbb{A}_j(1)$, \bar{G}_i and \bar{H}_i will be evaluated perturbatively within the gradient expansion (1.5). In section 4.2, we will consider another approximation—linearisation of the constitutive relation in the external electric field.

Within the gradient expansion, up to the order $\mathcal{O}(\partial^3)$, (1.11, 1.12) are

$$\begin{aligned} \vec{J} = & 12\kappa\mu_5 \vec{\mathbf{B}} + \vec{E} - \frac{\log 2}{2} \partial_t \vec{E} - \frac{\pi^2}{24} \partial_t^2 \vec{E} - \left(\frac{3}{2} \pi + 3 \log 2 \right) \kappa \partial_t \rho_5 \vec{\mathbf{B}} \\ & + 9\pi^2 \kappa^3 \rho_5 \left(\vec{\mathbf{B}} \times \vec{E} \right) \times \vec{E} + 12\#_1 \kappa \partial_t^2 \rho_5 \vec{\mathbf{B}} + \mathcal{O}(\partial^4), \end{aligned} \quad (1.13)$$

$$\begin{aligned} \vec{J}_5 = & 12\kappa\mu \vec{\mathbf{B}} - 36 \log 2 \kappa^2 \rho_5 \vec{\mathbf{B}} \times \vec{E} + \frac{3}{2} (\pi^2 + 3\pi \log 2 + 6 \log^2 2) \kappa^2 \partial_t \rho_5 \vec{\mathbf{B}} \times \vec{E} \\ & - \frac{3}{8} (48\mathcal{C} + \pi^2 - 12\pi \log 2) \kappa^2 \rho_5 \vec{\mathbf{B}} \times \partial_t \vec{E} + \mathcal{O}(\partial^4), \end{aligned} \quad (1.14)$$

where \mathcal{C} is a Catalan constant and $\#_1$ is known numerically only

$$\#_1 \approx 0.362. \quad (1.15)$$

Up to second order in derivatives $\mathcal{O}(\partial^2)$, the chemical potentials are⁴

$$\mu = 0 + \mathcal{O}(\partial^3), \quad \mu_5 = \frac{1}{2} \rho_5 + \frac{3}{2} (\pi - 2 \log 2) \kappa \vec{E} \cdot \vec{\mathbf{B}} + 18 (1 - 2 \log 2) \kappa^2 \rho_5 \mathbf{B}^2 + \mathcal{O}(\partial^3). \quad (1.16)$$

²We thank Dmitri Kharzeev for proposing us this study.

³In principle it is not excluded that the charge densities ρ, ρ_5 could be spatially-inhomogeneous. Yet such spatial inhomogeneity would render the derivative resummation highly complicated.

⁴While we suspect that the chemical potential μ is zero to all orders in the gradient expansion, we have not been able to prove that.

Evaluated on shell via (1.1), the axial current J_5^i is fully nonlinear in the amplitude of the electric field $\vec{E}(t)$, as clear from (1.14).

In the linearised regime, we assume the following scaling for the fields ρ_5 , \vec{E} and \vec{B}

$$\rho_5 \sim \mathcal{O}(\epsilon), \quad \vec{E}(t) \sim \mathcal{O}(\epsilon), \quad \vec{B} \sim \mathcal{O}(\epsilon^0), \quad (1.17)$$

which will be referred to as amplitude expansion. To linear order in ϵ , the vector/axial currents are

$$J^t = 0, \quad \vec{J} = \sigma_e \vec{E} + \tau_1 \kappa \rho_5 \vec{B} + \tau_2 \kappa^2 (\vec{E} \cdot \vec{B}) \vec{B}; \quad J_5^t = \rho_5, \quad \vec{J}_5 = 0, \quad (1.18)$$

where σ_e is a $q^2 = 0$ limit of the electric conductivity introduced in (1.2), while $\tau_{1,2}$ are new TCFs. As with other TCFs, they are functionals of time derivative operator and become functions of frequency ω in Fourier space,

$$\sigma_e[\partial_t] \longrightarrow \sigma_e(\omega), \quad \tau_{1,2}[\partial_t] \longrightarrow \tau_{1,2}(\omega). \quad (1.19)$$

Imposing the continuity equation (1.1), the electric current is put on-shell,

$$J^i = \sigma_{ij} E_j, \quad \sigma_{ij} = \underbrace{\sigma_e}_{\sigma_T} \left(\delta_{ij} - \frac{\mathbf{B}_i \mathbf{B}_j}{\mathbf{B}^2} \right) + \underbrace{\left[\sigma_e - \left(\frac{12}{i\omega} \tau_1 - \tau_2 \right) \kappa^2 \mathbf{B}^2 \right]}_{\sigma_L} \frac{\mathbf{B}_i \mathbf{B}_j}{\mathbf{B}^2}, \quad (1.20)$$

where the transverse conductivity σ_T is not affected by the magnetic field in contrast to the longitudinal conductivity σ_L which gets corrected by the magnetic field via the chiral anomaly. Particularly, in the DC limit

$$\sigma_L^0 = \frac{i}{\omega} 12 \kappa^2 \mathbf{B}^2 \tau_1^0 + [\sigma_e^0 + \kappa^2 \mathbf{B}^2 (\tau_2^0 - 12 \tau_1^1)], \quad (1.21)$$

where $\sigma_e^0 = 1$, and τ_1^0 and τ_2^0 are

$$\begin{aligned} \tau_1^0 &= \frac{\Gamma[3/4 - \sqrt{1 - 144 \kappa^2 \mathbf{B}^2}/4] \Gamma[3/4 + \sqrt{1 - 144 \kappa^2 \mathbf{B}^2}/4]}{3 \kappa^2 \mathbf{B}^2 \Gamma[1/4 - \sqrt{1 - 144 \kappa^2 \mathbf{B}^2}/4] \Gamma[1/4 + \sqrt{1 - 144 \kappa^2 \mathbf{B}^2}/4]} \\ &\longrightarrow 6 + 216 (1 - 2 \log 2) \kappa^2 \mathbf{B}^2 + \mathcal{O}(\mathbf{B}^4), \quad \text{as } \mathbf{B} \rightarrow 0, \end{aligned} \quad (1.22)$$

$$\tau_2^0 = 18 (\pi - 2 \log 2) + \#_2 \kappa^2 \mathbf{B}^2 + \mathcal{O}(\mathbf{B}^4), \quad \text{as } \mathbf{B} \rightarrow 0, \quad (1.23)$$

where $\Gamma[z]$ is a Gamma function, and $\#_2$ is known numerically only

$$\#_2 \approx -495.268. \quad (1.24)$$

τ_1^1 is the coefficient of $i\omega$ in hydrodynamic expansion of τ_1 . When the magnetic field is very strong, τ_1^0 and τ_2^0 behave similarly

$$\tau_1^0, \tau_2^0 \longrightarrow \frac{1}{\kappa \mathbf{B}}, \quad \text{as } \kappa \mathbf{B} \rightarrow \infty. \quad (1.25)$$

The result for τ_1^0 is in agreement with [62, 63]. Numerically computed \mathbf{B} -dependence of τ_1^1 and τ_2^0 and ω -dependence of τ_1, τ_2 will be presented in section 4.2.

In our calculation, $\text{Re}(\sigma_L^0)$ acquires negative correction due to magnetic field and eventually vanishes when the magnetic field gets large, see Figure 1. This is in contrast with many related studies of negative magnetoresistivity, the phenomenon of enhancement of longitudinal DC conductivity due to magnetic field [64–69]. However, taking strict DC limit in σ_L^0 is problematic due to the explicit $1/\omega$ divergence. The latter is frequently regularised by introduction of axial charge dissipation effects via shifting the frequency $\omega \rightarrow \omega + i/\tau_5$, where τ_5 corresponds to some relaxation time. The physics of this axial charge relaxation is beyond the scope of the present work. It was addressed within the holographic approach in [62, 70–72]. These studies primarily rely on the Kubo formula. Utilising the weak electric field approximation (1.17), [70] analytically evaluated the magnetic field dependence of the longitudinal conductivity σ_L in DC limit, while [62] calculated its ω -dependence. Back-reaction effects on σ_L were considered in [72]. Ref. [63] performed similar study focusing on time evolution of the induced vector current, given some specially chosen initial profile for the electric field.

While there is some overlap between our results and the literature, differences between the present study and those of [62, 70–72] must be clarified. All the studies [62, 70–72] focused on a weak electric field, in which the axial current vanishes. So, our nonlinear results and particularly the axial charge separation current (1.14) appear as new. As for the linearised setup (1.17), [62, 70–72] imposed the continuity equation and replaced the axial charge density ρ_5 in favour of the external electric and magnetic fields, so the vector current there is on-shell. This is in contrast to our off-shell formalism. As we argued in our previous publications [51, 53–56], only off-shell construction reveals transport properties of the system in full. Particularly, there are three independent TCFs (σ_e and $\tau_{1,2}$) in the constitutive relation (1.18), all of which we are able to determine separately, compared to only two independent conductivities in (1.20).

Another difference worth mentioning is that we explicitly trace all the effects in the induced current that arise from the relative angle between $\vec{E}(t)$ and \vec{B} fields. This is in contrast to [62, 63], which limited their study to the case of parallel fields only, primarily focusing on the longitudinal electric conductivity σ_L . By varying the relative angle between $\vec{E}(t)$ and \vec{B} fields, one can separate the anomaly induced effects (parametrised by τ_1 and τ_2) from the ones that are not related to the anomaly (σ_e).

The paper is structured as follows. In section 2 we present the holographic model and outline the strategy of deriving the boundary currents from solutions of the anomalous Maxwell equations in the bulk. Section 3 presents the first part of our study: CME/CSE with static but varying in space magnetic field. In section 4, CME/CSE in the presence of constant magnetic and time-varying electric fields are analysed. This study is further split into two subsections. Exploration of nonlinear phenomena in the induced vector/axial currents is done in 4.1. In section 4.2 we focus on the linearised regime (1.17) and calculate the dependence of AC conductivity on magnetic field. The last section 5 presents the conclusions.

2 The holographic model: $U(1)_V \times U(1)_A$

The holographic model is the $U(1)_V \times U(1)_A$ theory in the Schwarzschild- AdS_5 . The chiral anomaly of the boundary field theory is modelled via the gauge Chern-Simons terms in the bulk action

$$S = \int d^5x \sqrt{-g} \mathcal{L} + S_{\text{c.t.}}, \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}(F^V)_{MN}(F^V)^{MN} - \frac{1}{4}(F^a)_{MN}(F^a)^{MN} + \frac{\kappa \epsilon^{MNPQR}}{2\sqrt{-g}} \\ & \times [3A_M(F^V)_{NP}(F^V)_{QR} + A_M(F^a)_{NP}(F^a)_{QR}], \end{aligned} \quad (2.2)$$

and the counter-term action $S_{\text{c.t.}}$ is

$$S_{\text{c.t.}} = \frac{1}{4} \log r \int d^4x \sqrt{-\gamma} [(F^V)_{\mu\nu}(F^V)^{\mu\nu} + (F^a)_{\mu\nu}(F^a)^{\mu\nu}]. \quad (2.3)$$

ϵ^{MNPQR} is the Levi-Civita symbol with the convention $\epsilon^{txyz} = +1$, and the Levi-Civita tensor is $\epsilon^{MNPQR}/\sqrt{-g}$. Our choice for (2.3) is based on minimal subtraction, that is the counter-term does not make *finite* contribution to the boundary currents.

In the ingoing Eddington-Finkelstein coordinates, the spacetime metric is

$$ds^2 = g_{MN} dx^M dx^N = 2dt dr - r^2 f(r) dt^2 + r^2 \delta_{ij} dx^i dx^j, \quad (2.4)$$

where $f(r) = 1 - 1/r^4$, so that the Hawking temperature (identified as temperature of the boundary theory) is normalised to $\pi T = 1$. On the constant r hypersurface Σ , the induced metric $\gamma_{\mu\nu}$ is

$$ds^2|_{\Sigma} = \gamma_{\mu\nu} dx^\mu dx^\nu = -r^2 f(r) dt^2 + r^2 \delta_{ij} dx^i dx^j. \quad (2.5)$$

Equations of motion for V and A fields are

$$\text{dynamical equations: } EV^\mu = EA^\mu = 0, \quad (2.6)$$

$$\text{constraint equations: } EV^r = EA^r = 0, \quad (2.7)$$

where

$$EV^M \equiv \nabla_N (F^V)^{NM} + \frac{3\kappa \epsilon^{MNPQR}}{\sqrt{-g}} (F^a)_{NP} (F^V)_{QR}, \quad (2.8)$$

$$EA^M \equiv \nabla_N (F^a)^{NM} + \frac{3\kappa \epsilon^{MNPQR}}{2\sqrt{-g}} [(F^V)_{NP} (F^V)_{QR} + (F^a)_{NP} (F^a)_{QR}]. \quad (2.9)$$

The boundary currents are defined as

$$J^\mu \equiv \lim_{r \rightarrow \infty} \frac{\delta S}{\delta V_\mu}, \quad J_5^\mu \equiv \lim_{r \rightarrow \infty} \frac{\delta S}{\delta A_\mu}, \quad (2.10)$$

which, in terms of the bulk fields, are

$$\begin{aligned} J^\mu = & \lim_{r \rightarrow \infty} \sqrt{-\gamma} \left\{ (F^V)^{\mu M} n_M + \frac{6\kappa \epsilon^{M\mu NQR}}{\sqrt{-g}} n_M A_N (F^V)_{QR} - \tilde{\nabla}_\nu (F^V)^{\nu\mu} \log r \right\}, \\ J_5^\mu = & \lim_{r \rightarrow \infty} \sqrt{-\gamma} \left\{ (F^a)^{\mu M} n_M + \frac{2\kappa \epsilon^{M\mu NQR}}{\sqrt{-g}} n_M A_N (F^a)_{QR} - \tilde{\nabla}_\nu (F^a)^{\nu\mu} \log r \right\}, \end{aligned} \quad (2.11)$$

where n_M is the outpointing unit normal vector on the slice Σ , and $\tilde{\nabla}$ is compatible with the induced metric $\gamma_{\mu\nu}$.

The currents (2.10) are defined independently of the constraint equations (2.7). Throughout this work, the radial gauge $V_r = A_r = 0$ will be assumed. Consequently, in order to completely determine the boundary currents (2.11) it is sufficient to solve the dynamical equations (2.6) for the bulk gauge fields V_μ, A_μ only, leaving the constraints aside. The constraint equations (2.7) give rise to the continuity equations (1.1). In this way, the currents' constitutive relations to be derived below are off-shell.

It is useful to reexpress the currents (2.11) in terms of the coefficients of near boundary asymptotic expansion of the bulk gauge fields. Near $r = \infty$,

$$V_\mu = \mathcal{V}_\mu + \frac{V_\mu^{(1)}}{r} + \frac{V_\mu^{(2)}}{r^2} - \frac{2V_\mu^L}{r^2} \log r + \mathcal{O}\left(\frac{\log r}{r^3}\right), \quad A_\mu = \frac{A_\mu^{(2)}}{r^2} + \mathcal{O}\left(\frac{\log r}{r^3}\right), \quad (2.12)$$

where

$$V_\mu^{(1)} = \mathcal{F}_{t\mu}^V, \quad 4V_\mu^L = \partial^\nu \mathcal{F}_{\mu\nu}^V. \quad (2.13)$$

In (2.12) the constant term for A_μ is set to zero given that axial external fields are turned off in our present study. The holographic dictionary implies that \mathcal{V}_μ is a gauge potential of external electromagnetic fields \vec{E} and \vec{B} ,

$$E_i = \mathcal{F}_{it}^V = \partial_i \mathcal{V}_t - \partial_t \mathcal{V}_i, \quad B_i = \frac{1}{2} \epsilon_{ijk} \mathcal{F}_{jk}^V = \epsilon_{ijk} \partial_j \mathcal{V}_k. \quad (2.14)$$

When obtaining (2.12, 2.13), only the dynamical equations (2.6) were utilised. The near-boundary data $V_\mu^{(2)}$ and $A_\mu^{(2)}$ have to be determined by completely solving (2.6) from the horizon to the boundary. The currents (2.11) become

$$J^\mu = \eta^{\mu\nu} (2V_\nu^{(2)} + 2V_\nu^L + \eta^{\sigma t} \partial_\sigma \mathcal{F}_{t\nu}^V), \quad J_5^\mu = \eta^{\mu\nu} 2A_\nu^{(2)}. \quad (2.15)$$

The remainder of this section is to outline the strategy for deriving the constitutive relations for J^μ and J_5^μ . To this end, consider finite vector/axial charge densities exposed to external electromagnetic fields. Holographically, the charge densities and external fields are encoded in asymptotic behaviors of the bulk gauge fields. In the bulk, we will solve the dynamical equations (2.6) assuming some charge densities and external fields, but without specifying them explicitly.

Following [51] we start with the most general static and homogeneous profiles for the bulk gauge fields which solve the dynamical equations (2.6),

$$V_\mu = \mathcal{V}_\mu - \frac{\rho}{2r^2} \delta_{\mu t}, \quad A_\mu = -\frac{\rho_5}{2r^2} \delta_{\mu t}, \quad (2.16)$$

where $\mathcal{V}_\mu, \rho, \rho_5$ are all constants for the moment. Regularity requirement at $r = 1$ fixes one integration constant for each V_i and A_i . As explained below (2.13), the constant in A_μ is set to zero. Through (2.15), the boundary currents are

$$J^t = \rho, \quad J^i = 0; \quad J_5^t = \rho_5, \quad J_5^i = 0. \quad (2.17)$$

Hence, ρ and ρ_5 are identified as the vector/axial charge densities.

Next, following the idea of fluid/gravity correspondence [57], we promote $\mathcal{V}_\mu, \rho, \rho_5$ into arbitrary functions of the boundary coordinates

$$\mathcal{V}_\mu \rightarrow \mathcal{V}_\mu(x_\alpha), \quad \rho \rightarrow \rho(x_\alpha), \quad \rho_5 \rightarrow \rho_5(x_\alpha). \quad (2.18)$$

Then, (2.16) ceases to be a solution of the dynamical equations (2.6). To have them satisfied, suitable corrections in V_μ and A_μ have to be introduced:

$$V_\mu(r, x_\alpha) = \mathcal{V}_\mu(x_\alpha) - \frac{\rho(x_\alpha)}{2r^2} \delta_{\mu t} + \mathbb{V}_\mu(r, x_\alpha), \quad A_\mu(r, x_\alpha) = -\frac{\rho_5(x_\alpha)}{2r^2} \delta_{\mu t} + \mathbb{A}_\mu(r, x_\alpha), \quad (2.19)$$

where $\mathbb{V}_\mu, \mathbb{A}_\mu$ will be determined from solving (2.6). Appropriate boundary conditions have to be specified. First, \mathbb{V}_μ and \mathbb{A}_μ have to be regular over the whole integration interval of $r \in [1, \infty]$. Second, at the conformal boundary $r = \infty$, we require

$$\mathbb{V}_\mu \rightarrow 0, \quad \mathbb{A}_\mu \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty, \quad (2.20)$$

which amounts to fixing external gauge potentials to be \mathcal{V}_μ and zero (for the axial fields). Additional integration constants will be fixed by the Landau frame convention for the currents,

$$J^t = \rho(x_\alpha), \quad J_5^t = \rho_5(x_\alpha). \quad (2.21)$$

The Landau frame choice can be identified as a residual gauge fixing for the bulk fields.

The vector/axial chemical potentials are defined as

$$\begin{aligned} \mu &= V_t(r = \infty) - V_t(r = 1) = \frac{1}{2}\rho - \mathbb{V}_t(r = 1), \\ \mu_5 &= A_t(r = \infty) - A_t(r = 1) = \frac{1}{2}\rho_5 - \mathbb{A}_t(r = 1). \end{aligned} \quad (2.22)$$

Generically, μ, μ_5 are nonlinear functionals of densities and external fields.

In terms of \mathbb{V}_μ and \mathbb{A}_μ , the dynamical equations (2.6) are

$$0 = r^3 \partial_r^2 \mathbb{V}_t + 3r^2 \partial_r \mathbb{V}_t + r \partial_r \partial_k \mathbb{V}_k + 12\kappa \epsilon^{ijk} [\partial_r \mathbb{A}_i (\partial_j \mathcal{V}_k + \partial_j \mathbb{V}_k) + \partial_r \mathbb{V}_i \partial_j \mathbb{A}_k], \quad (2.23)$$

$$\begin{aligned} 0 &= (r^5 - r) \partial_r^2 \mathbb{V}_i + (3r^4 + 1) \partial_r \mathbb{V}_i + 2r^3 \partial_r \partial_t \mathbb{V}_i - r^3 \partial_r \partial_i \mathbb{V}_t + r^2 (\partial_t \mathbb{V}_i - \partial_i \mathbb{V}_t) \\ &\quad + r (\partial^2 \mathbb{V}_i - \partial_i \partial_k \mathbb{V}_k) - \frac{1}{2} \partial_i \rho + r^2 (\partial_t \mathcal{V}_i - \partial_i \mathcal{V}_t) + r (\partial^2 \mathcal{V}_i - \partial_i \partial_k \mathcal{V}_k) \\ &\quad + 12\kappa r^2 \epsilon^{ijk} \left(\frac{1}{r^3} \rho_5 \partial_j \mathcal{V}_k + \frac{1}{r^3} \rho_5 \partial_j \mathbb{V}_k + \partial_r \mathbb{A}_t \partial_j \mathcal{V}_k + \partial_r \mathbb{A}_t \partial_j \mathbb{V}_k \right) \\ &\quad - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{A}_j \left[(\partial_t \mathcal{V}_k - \partial_k \mathcal{V}_t) + (\partial_t \mathbb{V}_k - \partial_k \mathbb{V}_t) + \frac{1}{2r^2} \partial_k \rho \right] \\ &\quad - 12\kappa r^2 \epsilon^{ijk} \left\{ \partial_r \mathbb{V}_j \left[(\partial_t \mathbb{A}_k - \partial_k \mathbb{A}_t) + \frac{1}{2r^2} \partial_k \rho_5 \right] - \partial_j \mathbb{A}_k \left(\partial_r \mathbb{V}_t + \frac{1}{r^3} \rho \right) \right\}, \end{aligned} \quad (2.24)$$

$$0 = r^3 \partial_r^2 \mathbb{A}_t + 3r^2 \partial_r \mathbb{A}_t + r \partial_r \partial_k \mathbb{A}_k + 12\kappa \epsilon^{ijk} [\partial_r \mathbb{V}_i (\partial_j \mathcal{V}_k + \partial_j \mathbb{V}_k) + \partial_r \mathbb{A}_i \partial_j \mathbb{A}_k], \quad (2.25)$$

$$\begin{aligned}
0 = & (r^5 - r)\partial_r^2 \mathbb{A}_i + (3r^4 + 1)\partial_r \mathbb{A}_i + 2r^3 \partial_r \partial_t \mathbb{A}_i - r^3 \partial_r \partial_i \mathbb{A}_t + r^2 (\partial_t \mathbb{A}_i - \partial_i \mathbb{A}_t) \\
& + r(\partial^2 \mathbb{A}_i - \partial_i \partial_k \mathbb{A}_k) - \frac{1}{2} \partial_i \rho_5 + 12\kappa r^2 \epsilon^{ijk} (\partial_j \mathcal{V}_k + \partial_j \mathbb{V}_k) \left(\partial_r \mathbb{V}_t + \frac{1}{r^3} \rho \right) \\
& - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{V}_j \left[(\partial_t \mathcal{V}_k - \partial_k \mathcal{V}_t) + (\partial_t \mathbb{V}_k - \partial_k \mathbb{V}_t) + \frac{1}{2r^2} \partial_k \rho \right] \\
& - 12\kappa r^2 \epsilon^{ijk} \left\{ \partial_r \mathbb{A}_j \left[(\partial_t \mathbb{A}_k - \partial_k \mathbb{A}_t) + \frac{1}{2r^2} \partial_k \rho_5 \right] - \partial_j \mathbb{A}_k \left(\partial_r \mathbb{A}_t + \frac{1}{r^3} \rho_5 \right) \right\}.
\end{aligned} \tag{2.26}$$

In the following sections we will present solutions to the dynamical equations (2.23-2.26) under two setups discussed in the Introduction.

3 CME/CSE with time-independent inhomogeneous magnetic field

In this section we consider the case in which the magnetic field is the only external field that is turned on. The magnetic field is assumed to be varying in space, but it should be time-independent to avoid creating an electric field. There is no restriction on charge densities ρ, ρ_5 . From the general results (2.12, 2.13),

$$\mathbb{V}_t, \mathbb{A}_t \sim \mathcal{O}\left(\frac{\log r}{r^3}\right), \quad \mathbb{V}_i \sim \mathcal{O}\left(\frac{\log r}{r^2}\right), \quad \mathbb{A}_i \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad \text{as } r \rightarrow \infty. \tag{3.1}$$

In obtaining large r estimates for \mathbb{V}_t and \mathbb{A}_t , the frame convention (2.21) was used to fix the coefficients of $1/r^2$ in near-boundary expansion for V_t, A_t (thus those of \mathbb{V}_t and \mathbb{A}_t). The dynamical equations (2.23-2.26) could be put into integral forms

$$\begin{aligned}
\mathbb{V}_t(r) = & - \int_r^\infty \frac{dx}{x^3} \int_x^\infty \left\{ y \partial_y \partial_k \mathbb{V}_k + 12\kappa \epsilon^{ijk} [\partial_y \mathbb{A}_i (\partial_j \mathcal{V}_k + \partial_j \mathbb{V}_k) + \partial_y \mathbb{V}_i \partial_j \mathbb{A}_k] \right\} dy, \\
& \xrightarrow{r \rightarrow \infty} \mathcal{O}\left(\frac{\log r}{r^3}\right),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\mathbb{V}_i(r) = & \int_r^\infty \frac{-x dx}{x^4 - 1} \left\{ G_i(x) - \frac{1-x}{2x} \partial_i \rho - \partial_k \mathcal{F}_{ki}^V \log x - 12\kappa B_i \left(\mu_5 + \mathbb{A}_t - \frac{1}{2x^2} \rho_5 \right) \right\} \\
& \xrightarrow{r \rightarrow \infty} - \frac{\partial_i \rho}{4r^2} + \frac{1+2\log r}{4r^2} \partial_k \mathcal{F}_{ki}^V + \frac{6}{r^2} \kappa \mu_5 B_i - \frac{1}{2r^2} G_i(x=\infty) + \mathcal{O}\left(\frac{\log r}{r^3}\right),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\mathbb{A}_t(r) = & - \int_r^\infty \frac{dx}{x^3} \int_x^\infty \left\{ y \partial_y \partial_k \mathbb{A}_k + 12\kappa \epsilon^{ijk} [\partial_y \mathbb{V}_i (\partial_j \mathcal{V}_k + \partial_j \mathbb{V}_k) + \partial_y \mathbb{A}_i \partial_j \mathbb{A}_k] \right\} dy \\
& \xrightarrow{r \rightarrow \infty} \mathcal{O}\left(\frac{\log r}{r^3}\right),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\mathbb{A}_i(r) = & - \int_r^\infty \frac{x dx}{x^4 - 1} \left\{ H_i(x) - \frac{1-x}{2x} \partial_i \rho_5 - 12\kappa \mu B_i - 12\kappa B_i \left(\mathbb{V}_t - \frac{1}{2x^2} \rho \right) \right\} \\
& \xrightarrow{r \rightarrow \infty} - \frac{\partial_i \rho_5}{4r^2} + \frac{6}{r^2} \kappa \mu B_i - \frac{1}{2r^2} H_i(x=\infty) + \mathcal{O}\left(\frac{1}{r^3}\right),
\end{aligned} \tag{3.5}$$

where μ and μ_5 are the chemical potentials defined in (2.22). The frame convention (2.21) was utilised to fix integration constants, one for \mathbb{V}_t and one for \mathbb{A}_t . The functions $G_i(x)$

and $H_i(x)$ are

$$\begin{aligned}
G_i(x) = \int_1^x dy \left\{ -2y\partial_y\partial_t\mathbb{V}_i + y\partial_y\partial_i\mathbb{V}_t - (\partial_t\mathbb{V}_i - \partial_i\mathbb{V}_t) - \frac{1}{y}(\partial^2\mathbb{V}_i - \partial_i\partial_k\mathbb{V}_k) \right. \\
- 12\kappa\epsilon^{ijk}\partial_y\left(\mathbb{A}_t - \frac{1}{2y^2}\rho_5\right)\partial_j\mathbb{V}_k - 12\kappa\epsilon^{ijk}\partial_y\left(\mathbb{V}_t - \frac{1}{2y^2}\rho\right)\partial_j\mathbb{A}_k \\
+ 12\kappa\epsilon^{ijk}\partial_y\mathbb{A}_j\left[(\partial_t\mathbb{V}_k - \partial_k\mathbb{V}_t) + \frac{1}{2y^2}\partial_k\rho\right] + 12\kappa\epsilon^{ijk}\partial_y\mathbb{V}_j \\
\left. \times \left[(\partial_t\mathbb{A}_k - \partial_k\mathbb{A}_t) + \frac{1}{2y^2}\partial_k\rho_5\right] \right\}, \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
H_i(x) = \int_1^x dy \left\{ -2y\partial_y\partial_t\mathbb{A}_i + y\partial_y\partial_i\mathbb{A}_t - (\partial_t\mathbb{A}_i - \partial_i\mathbb{A}_t) - \frac{1}{y}(\partial^2\mathbb{A}_i - \partial_i\partial_k\mathbb{A}_k) \right. \\
- 12\kappa\epsilon^{ijk}\partial_y\left(\mathbb{V}_t - \frac{1}{2y^2}\rho\right)\partial_j\mathbb{V}_k - 12\kappa\epsilon^{ijk}\partial_y\left(\mathbb{A}_t - \frac{1}{2y^2}\rho_5\right)\partial_j\mathbb{A}_k \\
+ 12\kappa\epsilon^{ijk}\partial_y\mathbb{V}_j\left[(\partial_t\mathbb{V}_k - \partial_k\mathbb{V}_t) + \frac{1}{2y^2}\partial_k\rho\right] + 12\kappa\epsilon^{ijk}\partial_y\mathbb{A}_j \\
\left. \times \left[(\partial_t\mathbb{A}_k - \partial_k\mathbb{A}_t) + \frac{1}{2y^2}\partial_k\rho_5\right] \right\}. \tag{3.7}
\end{aligned}$$

Substituting near-boundary behavior (3.2-3.5) into (2.15) produces the results (1.3,1.4). G_i, H_i are functionals of ρ, ρ_5, \vec{B} .

We proceed with perturbative solution of the dynamical equations (2.23-2.26) within the boundary derivative expansion (1.5). Let introduce a formal expansion parameter λ via the scaling

$$\partial_\mu = (\partial_t, \partial_i) \longrightarrow (\lambda\partial_t, \lambda\partial_i), \tag{3.8}$$

the corrections \mathbb{V}_μ and \mathbb{A}_μ are expandable in λ ,

$$\mathbb{V}_\mu = \sum_{n=1}^{\infty} \lambda^n \mathbb{V}_\mu^{[n]}, \quad \mathbb{A}_\mu = \sum_{n=1}^{\infty} \lambda^n \mathbb{A}_\mu^{[n]}. \tag{3.9}$$

$\mathbb{V}_\mu^{[n]}$ and $\mathbb{A}_\mu^{[n]}$ up to $n = 2$ are listed below.

$$\mathbb{V}_t^{[1]} = \mathbb{A}_t^{[1]} = 0, \tag{3.10}$$

$$\mathbb{V}_i^{[1]} = -\frac{1}{8} \left[\log \frac{1+r^2}{(1+r)^2} - 2 \arctan(r) + \pi \right] \partial_i \rho + 3\kappa\rho_5 B_i \log \frac{1+r^2}{r^2}, \tag{3.11}$$

$$\mathbb{A}_i^{[1]} = \mathbb{V}_i^{[1]}(\rho \leftrightarrow \rho_5), \tag{3.12}$$

$$\begin{aligned}
\mathbb{V}_t^{[2]} = - \int_r^\infty \frac{dx}{x^3} \int_x^\infty dy \left\{ \frac{y \partial^2 \rho}{2(y^2+1)(y+1)} - \frac{y 6\kappa B_k \partial_k \rho_5}{(y^2+1)(y+1)} - \frac{72\kappa^2}{y(y^2+1)} \rho B^2 \right\} \\
\stackrel{r=1}{=} -\frac{1}{16} (\pi - 2 \log 2) \partial^2 \rho + \frac{3}{4} (\pi - 2 \log 2) \kappa B_k \partial_k \rho_5 - 18(1 - 2 \log 2) \kappa^2 \rho B^2, \tag{3.13}
\end{aligned}$$

$$\mathbb{A}_t^{[2]} = \mathbb{V}_t^{[2]}(\rho \leftrightarrow \rho_5), \tag{3.14}$$

$$\begin{aligned}\mathbb{V}_i^{[2]} = & b_0 \partial_k \mathcal{F}_{ki}^V + b_1 \partial_t \partial_i \rho + b_2 6\kappa \partial_t \rho_5 B_i + b_3 36\kappa^2 \epsilon^{ijk} [\rho_5 \partial_j (\rho_5 B_k) + \rho \partial_j (\rho B_k)] \\ & + b_4 36\kappa^2 \epsilon^{ijk} (\rho B_j \partial_k \rho + \rho_5 B_j \partial_k \rho_5),\end{aligned}\quad (3.15)$$

$$\begin{aligned}\mathbb{A}_i^{[2]} = & b_1 \partial_t \partial_i \rho_5 + b_2 6\kappa \partial_t \rho B_i + b_3 36\kappa^2 \epsilon^{ijk} [\rho \partial_j (\rho_5 B_k) + \rho_5 \partial_j (\rho B_k)] + b_4 36\kappa^2 \epsilon^{ijk} \\ & \times (\rho_5 B_j \partial_k \rho + \rho B_j \partial_k \rho_5),\end{aligned}\quad (3.16)$$

where

$$b_0 = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x \frac{dy}{y}, \quad (3.17)$$

$$b_1 = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x dy \left\{ \frac{y}{(y^2+1)(y+1)} - \frac{1}{8} \left[\log \frac{1+y^2}{(1+y)^2} - 2 \arctan(y) + \pi \right] \right\}, \quad (3.18)$$

$$b_2 = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x dy \left\{ -\frac{2}{y^2+1} + \frac{1}{2} \log \frac{1+y^2}{y^2} \right\}, \quad (3.19)$$

$$b_3 = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x \frac{1}{y^3} \log \frac{1+y^2}{y^2} dy, \quad (3.20)$$

$$b_4 = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x \frac{dy}{y^3(y^2+1)}. \quad (3.21)$$

Substituting the first order solutions (3.10-3.12) into (3.6,3.7), we obtain the results (1.6,1.7). Meanwhile, the second order results (3.13,3.14) give rise to the expansion of the chemical potentials (1.8). In principle, the second order results (3.13-3.16) could be inserted into (3.6,3.7), producing derivative expansion for $G_i(x=\infty)$ and $H_i(x=\infty)$ up to third order. However, at third order $\mathcal{O}(\partial^3)$, computing G_i, H_i becomes quite involved. So, at third order $\mathcal{O}(\partial^3)$ we decided to track only linear in ρ, ρ_5 terms. As a result, we are able to identify the first anomalous correction to the diffusion constant \mathcal{D}_0 due to magnetic field. The final expressions are

$$\begin{aligned}G_i^{[3]}(x=\infty) = & \frac{\pi^2}{48} \partial_t^2 \partial_i \rho + \frac{1}{16} (\pi - 2 \log 2) \partial^2 \partial_i \rho + 12 \#_1 \kappa \partial_t^2 \rho_5 B_i - \frac{\pi^2}{8} \kappa \\ & \times [\partial^2 (\rho_5 B_i) - \partial_i \partial_k (\rho_5 B_k)] + \frac{3}{4} (\pi - 2 \log 2) \kappa \partial_i (B_k \partial_k \rho_5) \\ & + \underline{18(1 - 2 \log 2) \kappa^2 B^2 \partial_i \rho} + 18(1 - 2 \log 2) \kappa^2 \rho \partial_i B^2 + \mathcal{O}(\rho^2, \rho_5^2, \rho \rho_5),\end{aligned}\quad (3.22)$$

$$H_i^{[3]}(x=\infty) = G_i^{[3]}(x=\infty) (\rho \leftrightarrow \rho_5), \quad (3.23)$$

where $\#_1$ in (3.22) is given by the integral

$$\#_1 \equiv \frac{1}{2} \int_1^\infty dy [2y \partial_y b_2(y) + b_2(y)] \approx 0.362. \quad (3.24)$$

The underlined term in (3.22) is a $\kappa^2 B^2$ -correction to the diffusion constant.

Our results for J^μ and J_5^μ can be used to explore dispersion relations for free modes propagating in the chiral medium. We consider a constant magnetic field only. Let us take a plane wave ansatz for the vector/axial charge densities

$$\rho = \delta \rho \exp(-i\omega t + \vec{q} \cdot \vec{x}), \quad \rho_5 = \delta \rho_5 \exp(-i\omega t + \vec{q} \cdot \vec{x}). \quad (3.25)$$

Then the continuity equation (1.1) becomes

$$a\delta\rho + b\delta\rho_5 = 0, \quad b\delta\rho + a\delta\rho_5 = 0, \quad (3.26)$$

which has a nontrivial solution when and only when

$$a^2 = b^2 \implies a = \pm b, \quad (3.27)$$

where

$$a = -i\omega + \frac{1}{2}q^2 + 9(\pi - 2\log 2)\kappa^2(\vec{q} \cdot \vec{\mathbf{B}})^2 + 216(1 - 2\log 2)\kappa^3\mathbf{B}^2 i\vec{q} \cdot \vec{\mathbf{B}} + \frac{\pi}{8}i\omega q^2 - \frac{\pi^2}{48}\omega^2 q^2 - \frac{1}{16}(\pi - 2\log 2)q^4 + 18(1 - 2\log 2)\kappa^2\mathbf{B}^2 q^2, \quad (3.28)$$

$$b = 6\kappa i\vec{q} \cdot \vec{\mathbf{B}} - \frac{3}{4}(\pi - 2\log 2)\kappa q^2 \vec{q} \cdot \vec{\mathbf{B}} - \left(\frac{3}{2}\pi + 3\log 2\right)\kappa\omega \vec{q} \cdot \vec{\mathbf{B}} + 12\#_1\kappa\omega^2 i\vec{q} \cdot \vec{\mathbf{B}} + \frac{3}{4}(\pi - 2\log 2)\kappa q^2 \vec{q} \cdot \vec{\mathbf{B}}. \quad (3.29)$$

Solving (3.27) leads to the \mathbf{B} -corrected dispersion relation, as summarised in (1.10).

4 CME/CSE with constant magnetic and time-dependent electric fields

Creating systems with chiral imbalance ($\mu_5 \neq 0$) experimentally is problematic. In this section we consider a special setup in which the axial chemical potential μ_5 is not imposed externally but rather induced dynamically through chiral anomaly. This setup is of particular interest due to intriguing possibility for it to be realised experimentally in chiral condensed matter systems. Consider a constant magnetic field $\vec{\mathbf{B}}$ and a time-dependent homogeneous electric field $\vec{E}(t)$. We also assume the charge densities to be spatially-homogeneous as well⁵. The continuity equation (1.1) degenerates to

$$\partial_t J^t = 0, \quad \partial_t J_5^t = 12\kappa \vec{E} \cdot \vec{\mathbf{B}}, \quad (4.1)$$

which implies that the vector charge density is constant while the axial charge density has nontrivial time dependence inherited from $\vec{E}(t)$. The setup under consideration is

$$\rho = 0, \quad \rho_5 = \rho_5(t), \quad \vec{E} = \vec{E}(t), \quad \vec{\mathbf{B}} = \text{constant}. \quad (4.2)$$

Under the frame convention (2.21), the corrections \mathbb{V}_μ and \mathbb{A}_μ of (2.19) depend on r and t only. As a result, the dynamical equations (2.23-2.26) are reduced to

$$0 = r^3 \partial_r^2 \mathbb{V}_t + 3r^2 \partial_r \mathbb{V}_t + 12\kappa \partial_r \mathbb{A}_k \mathbf{B}_k, \quad (4.3)$$

$$0 = (r^5 - r) \partial_r^2 \mathbb{V}_i + (3r^4 + 1) \partial_r \mathbb{V}_i + 2r^3 \partial_r \partial_t \mathbb{V}_i + r^2 \partial_t \mathbb{V}_i - r^2 E_i + 12\kappa r^2 \partial_r \mathbb{A}_t \mathbf{B}_i + \frac{12}{r} \kappa \rho_5 \mathbf{B}_i - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{A}_j (\partial_t \mathbb{V}_k - E_k) - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{V}_j \partial_t \mathbb{A}_k, \quad (4.4)$$

$$0 = r^3 \partial_r^2 \mathbb{A}_t + 3r^2 \partial_r \mathbb{A}_t + 12\kappa \partial_r \mathbb{V}_k \mathbf{B}_k, \quad (4.5)$$

$$0 = (r^5 - r) \partial_r^2 \mathbb{A}_i + (3r^4 + 1) \partial_r \mathbb{A}_i + 2r^3 \partial_r \partial_t \mathbb{A}_i + r^2 \partial_t \mathbb{A}_i + 12\kappa r^2 \partial_r \mathbb{V}_t \mathbf{B}_i - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{V}_j (\partial_t \mathbb{V}_k - E_k) - 12\kappa r^2 \epsilon^{ijk} \partial_r \mathbb{A}_j \partial_t \mathbb{A}_k. \quad (4.6)$$

⁵While from the continuity equation (1.1) the charge densities can still have a nontrivial spatial-dependence, we found that such spatial inhomogeneity of the charge densities would make the gradient resummation out of control.

4.1 Non-linear phenomena: general analysis and derivative expansion

The dynamical equations (4.3-4.6) can be put into integral forms

$$\mathbb{V}_t = 12\kappa \int_r^\infty \frac{dx}{x^3} \mathbb{A}_k \mathbf{B}_k \xrightarrow{r \rightarrow \infty} \mathcal{O}\left(\frac{1}{r^3}\right), \quad (4.7)$$

$$\begin{aligned} \mathbb{V}_i = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x dy \left\{ 2y\partial_y\partial_t\mathbb{V}_i + \partial_t\mathbb{V}_i - E_i + 12\kappa\partial_y\left(\mathbb{A}_t - \frac{1}{2y^2}\rho_5\right)\mathbf{B}_i \right. \\ \left. - 12\kappa\epsilon^{ijk}\partial_y\mathbb{A}_j(-E_k + \partial_t\mathbb{V}_k) - 12\kappa\epsilon^{ijk}\partial_y\mathbb{V}_j\partial_t\mathbb{A}_k \right\} \\ \xrightarrow{r \rightarrow \infty} \frac{1+2\log r}{4r^2}\partial_tE_i - \left(\frac{1}{r} - \frac{1}{2r^2}\right)E_i + \frac{6}{r^2}\kappa\mu_5\mathbf{B}_i - \frac{6}{r^2}\kappa\epsilon^{ijk}\mathbb{A}_j(1)E_k \\ + \frac{1}{2r^2}\overline{G}_i(x=\infty) + \mathcal{O}\left(\frac{1}{r^3}\right), \end{aligned} \quad (4.8)$$

$$\mathbb{A}_t = 12\kappa \int_r^\infty \frac{dx}{x^3} \mathbb{V}_k \mathbf{B}_k \xrightarrow{r \rightarrow \infty} \mathcal{O}\left(\frac{1}{r^3}\right), \quad (4.9)$$

$$\begin{aligned} \mathbb{A}_i = \int_r^\infty \frac{xdx}{x^4-1} \int_1^x dy \left\{ 2y\partial_y\partial_t\mathbb{A}_i + \partial_t\mathbb{A}_i + 12\kappa\partial_y\mathbb{V}_t\mathbf{B}_i + 12\kappa\epsilon^{ijk}\partial_y\mathbb{V}_jE_k \right. \\ \left. - 12\kappa\epsilon^{ijk}\partial_y\mathbb{V}_j\partial_t\mathbb{V}_k - 12\kappa\epsilon^{ijk}\partial_y\mathbb{A}_j\partial_t\mathbb{A}_k \right\} \\ \xrightarrow{r \rightarrow \infty} \frac{6}{r^2}\kappa\mu\mathbf{B}_i - \frac{6}{r^2}\kappa\epsilon^{ijk}\mathbb{V}_j(1)E_k + \frac{1}{2r^2}\overline{H}_i(x=\infty) + \mathcal{O}\left(\frac{1}{r^3}\right), \end{aligned} \quad (4.10)$$

where μ, μ_5 are defined via (2.22). \overline{G}_i and \overline{H}_i are

$$\overline{G}_i(x) = \int_1^x dy \left\{ 2y\partial_y\partial_t\overline{\mathbb{V}}_i + \partial_t\overline{\mathbb{V}}_i - 12\kappa\epsilon^{ijk}(\partial_y\mathbb{A}_j\partial_t\mathbb{V}_k + \partial_y\mathbb{V}_j\partial_t\mathbb{A}_k) \right\}, \quad (4.11)$$

$$\overline{H}_i(x) = \int_1^x dy \left\{ 2y\partial_y\partial_t\mathbb{A}_i + \partial_t\mathbb{A}_i - 12\kappa\epsilon^{ijk}(\partial_y\mathbb{V}_j\partial_t\mathbb{V}_k + \partial_y\mathbb{A}_j\partial_t\mathbb{A}_k) \right\}, \quad (4.12)$$

where $\overline{\mathbb{V}}_i = \mathbb{V}_i + E_i/r$.

From (2.15), the near-boundary expansion in (4.7-4.10) is translated into boundary currents (1.11, 1.12). For generic boundary conditions, the quantities $\mathbb{V}_i(1)$, $\mathbb{A}_i(1)$, $\overline{G}_i(x=\infty)$ and $\overline{H}_i(x=\infty)$ cannot be computed analytically. To proceed, we perturbatively solve the dynamical equations (4.3-4.6) under the derivative expansion (1.5), as done in section 3. Up to second order $\mathcal{O}(\partial^2)$, the corrections \mathbb{V}_μ and \mathbb{A}_μ are

$$\mathbb{V}_t = \mathcal{O}(\partial^3), \quad (4.13)$$

$$\mathbb{A}_t = a_0(r)12\kappa\vec{E} \cdot \vec{\mathbf{B}} - \frac{18}{r^2} \left[1 - (1+r^2)\log\frac{1+r^2}{r^2} \right] \kappa^2\rho_5\mathbf{B}^2 + \mathcal{O}(\partial^3), \quad (4.14)$$

$$\begin{aligned} \mathbb{V}_i = -\frac{1}{4} \left[\log\frac{(1+r)^2}{1+r^2} - 2\arctan(r) + \pi \right] E_i + 3\log\frac{1+r^2}{r^2}\kappa\rho_5\mathbf{B}_i + a_1(r)\partial_tE_i \\ + b_2(r)6\kappa\partial_t\rho_5\mathbf{B}_i + \mathcal{O}(\partial^3), \end{aligned} \quad (4.15)$$

$$\mathbb{A}_i = a_2(r)72\kappa^2\rho_5\epsilon^{ijk}\mathbf{B}_jE_k + \mathcal{O}(\partial^3), \quad (4.16)$$

where

$$a_0(r) = -\frac{1}{8r^2} \left\{ (r^2 + 1) \left(2 \operatorname{arccot}(r) - \log \frac{1+r^2}{r^2} \right) + 2(r^2 - 1) \log \frac{r}{1+r} \right\}, \quad (4.17)$$

$$a_1(r) = -\int_r^\infty \frac{xdx}{x^4-1} \int_1^x dy \left\{ -\frac{2y^2}{1+y^2} + \frac{1}{4} \left[\log \frac{(1+y)^2}{1+y^2} - 2 \arctan(y) + \pi \right] \right\}, \quad (4.18)$$

$$a_2(r) = -\int_r^\infty \frac{xdx}{x^4-1} \int_1^x \frac{dy}{y(y^2+1)}, \quad (4.19)$$

and b_2 is presented in (3.19). Substituting the perturbative solutions (4.13-4.16) into (4.11,4.12), we deduce the boundary currents up to third order in the derivative counting, see (1.13, 1.14). Meanwhile, from (2.22), the solutions (4.13,4.14) give rise to expansion for the chemical potentials, see (1.16).

4.2 Linear in \vec{E} phenomena

In the previous subsection we focused on hydrodynamic regime, in which we were able to identify some non-linear phenomena. Below, we proceed with an alternative approximation, that is the weak electric field approximation (1.17):

$$\rho_5(t) \sim \mathcal{O}(\epsilon), \quad \vec{E}(t) \sim \mathcal{O}(\epsilon), \quad \vec{\mathbf{B}} \sim \mathcal{O}(\epsilon^0). \quad (4.20)$$

The scaling of ρ_5 follows from the continuity equation (4.1). Both corrections \mathbb{V}_μ and \mathbb{A}_μ are of order $\mathcal{O}(\epsilon)$ too. The dynamical equations (4.3-4.6) get further simplified

$$0 = r^3 \partial_r^2 \mathbb{V}_t + 3r^2 \partial_r \mathbb{V}_t + 12\kappa \partial_r \mathbb{A}_k \mathbf{B}_k, \quad (4.21)$$

$$0 = (r^5 - r) \partial_r^2 \mathbb{V}_i + (3r^4 + 1) \partial_r \mathbb{V}_i + 2r^3 \partial_r \partial_t \mathbb{V}_i + r^2 \partial_t \mathbb{V}_i - r^2 E_i + 12\kappa r^2 \left(\partial_r \mathbb{A}_t + \frac{\rho_5}{r^3} \right) \mathbf{B}_i, \quad (4.22)$$

$$0 = r^3 \partial_r^2 \mathbb{A}_t + 3r^2 \partial_r \mathbb{A}_t + 12\kappa \partial_r \mathbb{V}_k \mathbf{B}_k, \quad (4.23)$$

$$0 = (r^5 - r) \partial_r^2 \mathbb{A}_i + (3r^4 + 1) \partial_r \mathbb{A}_i + 2r^3 \partial_r \partial_t \mathbb{A}_i + r^2 \partial_t \mathbb{A}_i + 12\kappa r^2 \partial_r \mathbb{V}_t \mathbf{B}_i. \quad (4.24)$$

Integrating (4.21,4.23) over r once, we get

$$\partial_r \mathbb{V}_t = -\frac{12\kappa}{r^3} \mathbb{A}_k \mathbf{B}_k, \quad \partial_r \mathbb{A}_t = -\frac{12\kappa}{r^3} \mathbb{V}_k \mathbf{B}_k, \quad (4.25)$$

where the frame convention (2.21) was used to fix the integration constant. (4.25) makes it possible to decouple $\mathbb{V}_i, \mathbb{A}_i$ from $\mathbb{V}_t, \mathbb{A}_t$. Consequently, (4.22,4.24) become

$$0 = (r^5 - r) \partial_r^2 \mathbb{V}_i + (3r^4 + 1) \partial_r \mathbb{V}_i + 2r^3 \partial_r \partial_t \mathbb{V}_i + r^2 \partial_t \mathbb{V}_i - r^2 E_i + \frac{12\kappa}{r} \mathbf{B}_i (\rho_5 - 12\kappa \mathbb{V}_k \mathbf{B}_k), \quad (4.26)$$

$$0 = (r^5 - r) \partial_r^2 \mathbb{A}_i + (3r^4 + 1) \partial_r \mathbb{A}_i + 2r^3 \partial_r \partial_t \mathbb{A}_i + r^2 \partial_t \mathbb{A}_i - \frac{144}{r} \kappa^2 \mathbf{B}_i (\mathbb{A}_k \mathbf{B}_k). \quad (4.27)$$

Homogeneity property of (4.27), combined with the regularity requirement at $r = 1$ and vanishing boundary condition at $r = \infty$ for \mathbb{A}_i , fixes $\mathbb{A}_i = 0$ completely. From (4.25), $\mathbb{V}_t = 0$. That is,

$$\mathbb{V}_t = \mathbb{A}_i = 0. \quad (4.28)$$

Therefore, at order $\mathcal{O}(\epsilon)$, the axial current $\vec{J}_5 = 0$. This is in contrast with the nonlinear analysis of section 4.1.

\mathbb{V}_i is decomposed as⁶

$$\mathbb{V}_i = C_1 E_i + C_2 \kappa \rho_5 \mathbf{B}_i + C_3 \kappa^2 (\vec{E} \cdot \vec{\mathbf{B}}) \mathbf{B}_i, \quad (4.29)$$

where

$$C_i = C_i(r, \partial_t) \rightarrow C_i(r, \omega), \quad i = 1, 2, 3. \quad (4.30)$$

The decomposition coefficients C_i 's satisfy partially decoupled ordinary differential equations (ODEs),

$$0 = (r^5 - r) \partial_r^2 C_1 + (3r^4 + 1) \partial_r C_1 - 2i\omega r^3 \partial_r C_1 - i\omega r^2 C_1 - r^2, \quad (4.31)$$

$$0 = (r^5 - r) \partial_r^2 C_2 + (3r^4 + 1) \partial_r C_2 - 2i\omega r^3 \partial_r C_2 - i\omega r^2 C_2 + \frac{12}{r} (1 - 12\kappa^2 \mathbf{B}^2 C_2), \quad (4.32)$$

$$0 = (r^5 - r) \partial_r^2 C_3 + (3r^4 + 1) \partial_r C_3 - 2i\omega r^3 \partial_r C_3 - i\omega r^2 C_3 - \frac{144}{r} (C_1 + \kappa^2 \mathbf{B}^2 C_3). \quad (4.33)$$

While C_1 does not feel the effect of magnetic field, $C_{2,3}$ have nontrivial dependence on the magnetic field via $\kappa^2 \mathbf{B}^2$.

Near $r = \infty$, pre-asymptotic expansions of C_i 's are

$$C_1 \rightarrow -\frac{1}{r} + \frac{c_1}{r^2} - \frac{i\omega \log r}{2r^2} + \mathcal{O}\left(\frac{\log r}{r^3}\right), \quad C_2 \rightarrow \frac{c_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad C_3 \rightarrow \frac{c_3}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (4.34)$$

where c_i 's are boundary data and have to be fixed through full solution of (4.31-4.33) from the horizon $r = 1$ to the conformal boundary $r = \infty$. From (2.15), the conductivities of (1.18) are determined by the boundary data c_i 's,

$$\sigma_e = 2c_1 - \frac{1}{2}i\omega, \quad \tau_1 = 2c_2, \quad \tau_2 = 2c_3. \quad (4.35)$$

The ODE for C_1 was solved in [51]. The conductivity σ_e , which is computed from C_1 was completely determined and explored in [51], while only $q = 0$ limit enters into our current study (the results are quoted below). We therefore focus on the remaining two conductivities τ_1, τ_2 , both induced by the chiral anomaly. As is obvious from (4.31-4.33), τ_1, τ_2 depend on the magnetic field via $\kappa^2 \mathbf{B}^2$.

Using the continuity equation (4.1), the constitutive relations (1.18) are put into a linear response form, from which on-shell current-current correlators can be read off. Since the electric field is the only external perturbation that is turned on, it is possible to compute only a subset of all two-point correlators in the theory,

$$\langle J^i J^j \rangle = \underbrace{i\omega \sigma_e}_{G^T} \left(\delta_{ij} - \frac{\mathbf{B}_i \mathbf{B}_j}{\mathbf{B}^2} \right) + \underbrace{[i\omega \sigma_e - (12\tau_1 - i\omega \tau_2) \kappa^2 \mathbf{B}^2]}_{G^L} \frac{\mathbf{B}_i \mathbf{B}_j}{\mathbf{B}^2}, \quad (4.36)$$

$$\langle J_5^t J^i \rangle = -12\kappa \mathbf{B}_i, \quad (4.37)$$

⁶In the decomposition for \mathbb{V}_i , one could have included a term $C_4 \vec{E} \times \vec{\mathbf{B}}$. However, the coefficient C_4 would satisfy a homogeneous ODE. Under the same arguments leading to $\mathbb{A}_i = 0$, C_4 has to be zero too.

$$\langle J^t J^i \rangle = \langle J_5^i J^j \rangle = 0, \quad (4.38)$$

where $\langle J^i J^j \rangle$ is split into transverse (G^T) and longitudinal (G^L) components with respect to the direction of $\vec{\mathbf{B}}$. To determine the remaining current-current correlators we would have to introduce additional field perturbations, particularly an axial external field, which is beyond the scope of this paper.

To evaluate the TCFs τ_1, τ_2 , we have to completely solve ODEs (4.31-4.33). We first analytically solve them when $\omega = 0$. As a result, the DC limits τ_1^0 (for arbitrary \mathbf{B}) and τ_2^0 (up to leading \mathbf{B}^2 -correction) are known analytically, as quoted in (1.22,1.23). The coefficient $\#_2$ in (1.23) is

$$\#_2 \equiv \int_1^\infty \frac{dr}{r^3} \left\{ \int_r^\infty \frac{72^2 x dx}{x^4 - 1} \int_1^x \frac{dy}{y} \left[\log \frac{(1+y)^2}{1+y^2} - 2 \arctan(y) + \pi \right] \right\} \approx -495.268. \quad (4.39)$$

For illustration, in figure 1 we show $\kappa\mathbf{B}$ -dependence of τ_1^0, τ_2^0 (divided by 5 to match scales), τ_1^1 and $\text{Re}(\sigma_L^0)$. The behaviour of $\text{Re}(\sigma_L^0)$ agrees with that of [62].

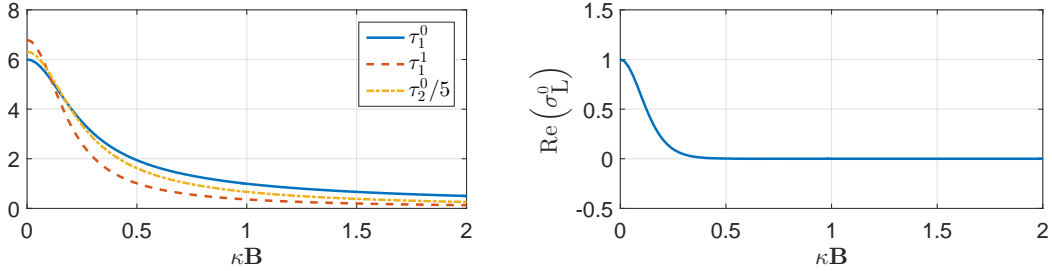


Figure 1. DC conductivities $\tau_1^0, \tau_2^0/5, \tau_1^1$, and $\text{Re}(\sigma_L^0)$ as function of $\kappa\mathbf{B}$.

Strong magnetic field limit of τ_1^0, τ_2^0 was also computed, and the results are summarised in (1.25). In the DC limit $\omega \rightarrow 0$, when the magnetic field is very strong the on-shell vector current behaves as

$$J^i \rightarrow -12\kappa\mathbf{B}\mathcal{V}_i, \quad (4.40)$$

which is in agreement with [63]. When $\omega \rightarrow 0$ (DC limit), the current-current correlator is dominated by the chiral anomaly induced effects $\sim \tau_1^0$. The DC limit is of interest for experiments with electric fields turned on adiabatically, such as the ones considered in [63].

For arbitrary ω , we resort to numerical methods and solve ODEs (4.31-4.33) for representative values of $\kappa\mathbf{B}$. The numerical procedure is identical to that of [52] and for all the numerical details we refer the reader to this publication. In Figure 2 we show ω -dependence for τ_1 and τ_2 for sample choices of $\kappa\mathbf{B}$. In Figure 3 we plot the normalised TCFs τ_1/τ_1^0 and τ_2/τ_2^0 . Overall, τ_1 and τ_2 display quite similar dependence on the frequency ω . After some oscillations, both τ_1 and τ_2 approach zero asymptotically.

Approach to the asymptotic regime, however, depends on strength of the magnetic field. When $\kappa\mathbf{B}$ is increased, the asymptotic behaviour is delayed towards larger ω . What is more intriguing is that increasing $\kappa\mathbf{B}$ renders τ_1 and τ_2 to develop a resonance-like enhancement at finite ω . This could be an interesting experimentally observable feature. For very strong

magnetic fields $\kappa\mathbf{B} \rightarrow \infty$, the chiral anomaly-induced effects would be pushed to the UV, corresponding to early time effects, such as in [63].

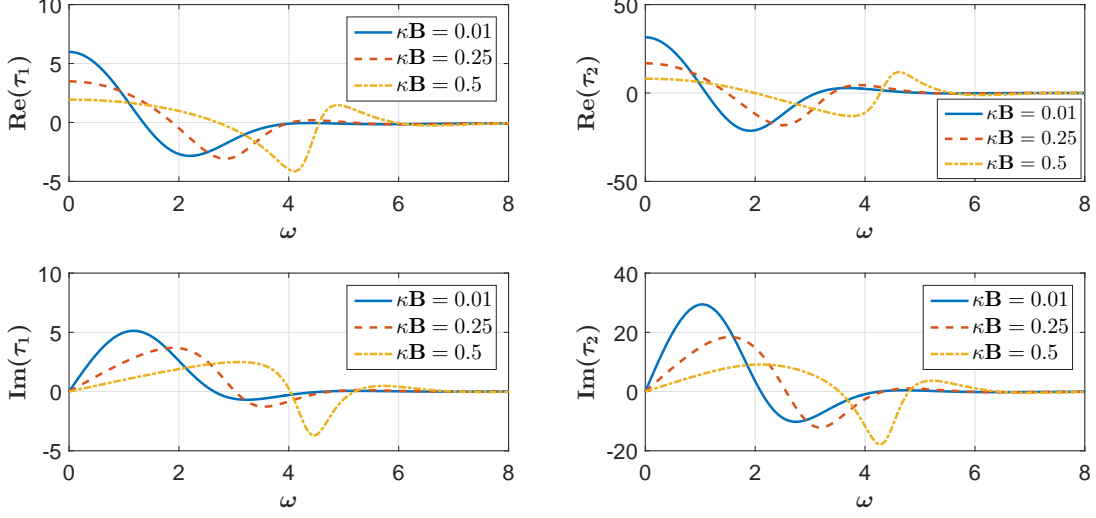


Figure 2. AC conductivities τ_1 and τ_2 for different values of $\kappa\mathbf{B}$.

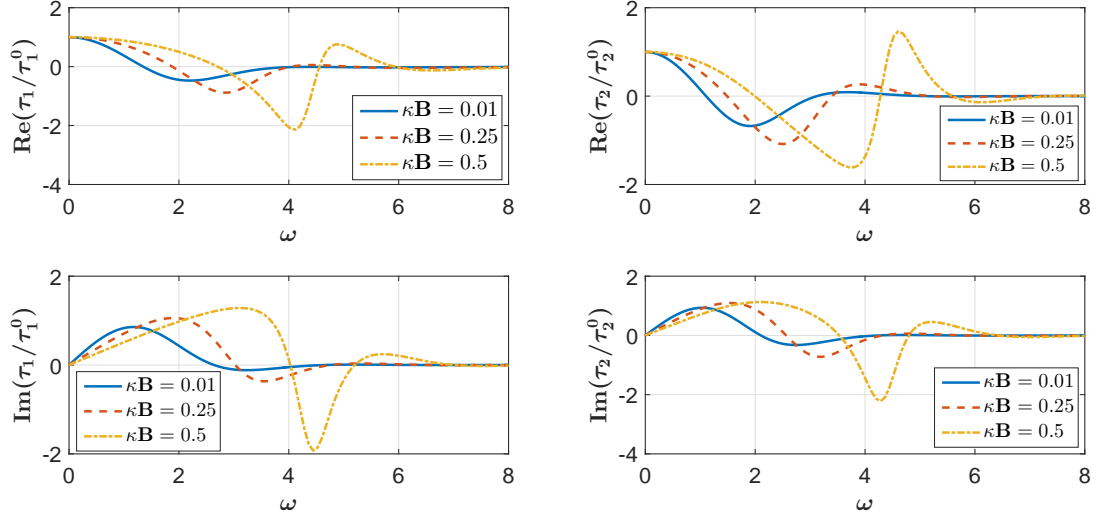


Figure 3. Normalised AC conductivities τ_1/τ_1^0 and τ_2/τ_2^0 for different values of $\kappa\mathbf{B}$.

In Figure 4 we show two-point correlators $G^{\text{T,L}}$ for different choices of $\kappa\mathbf{B}$. However, it is difficult to appreciate the anomaly induced effects from Figure 4 because in the correlators they get mixed with non-anomalous ones. To illuminate $\kappa\mathbf{B}$ -correction to G^{L} , in Figure 5 we plot the difference $\delta G^{\text{L}} = G^{\text{L}} - G^{\text{T}}$. From these plots, the effect of chiral anomaly on the induced vector current is seen more clearly. We again notice a remarkable relative enhancement at intermediate values of ω .

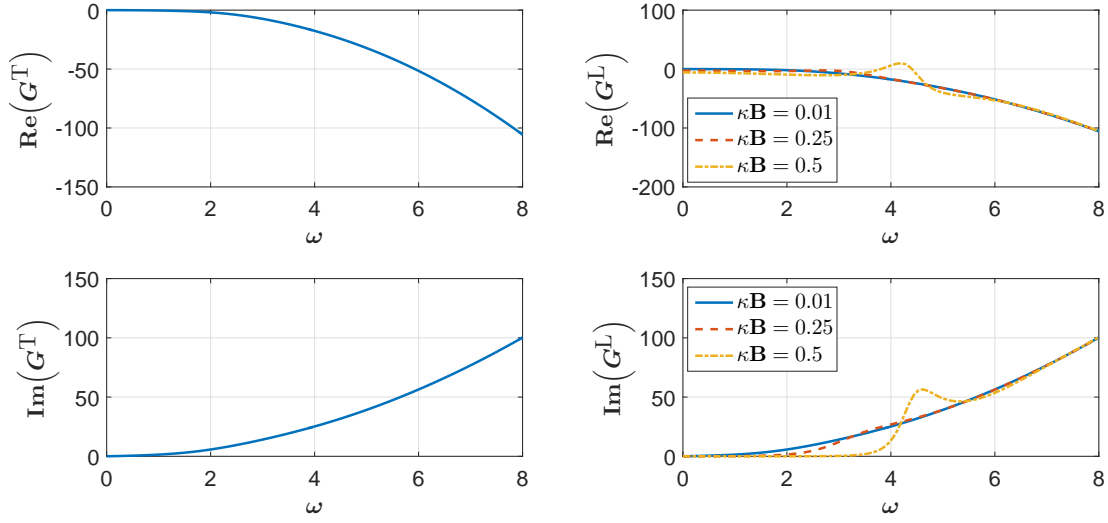


Figure 4. Current-current correlators G^T (left) and G^L (right).

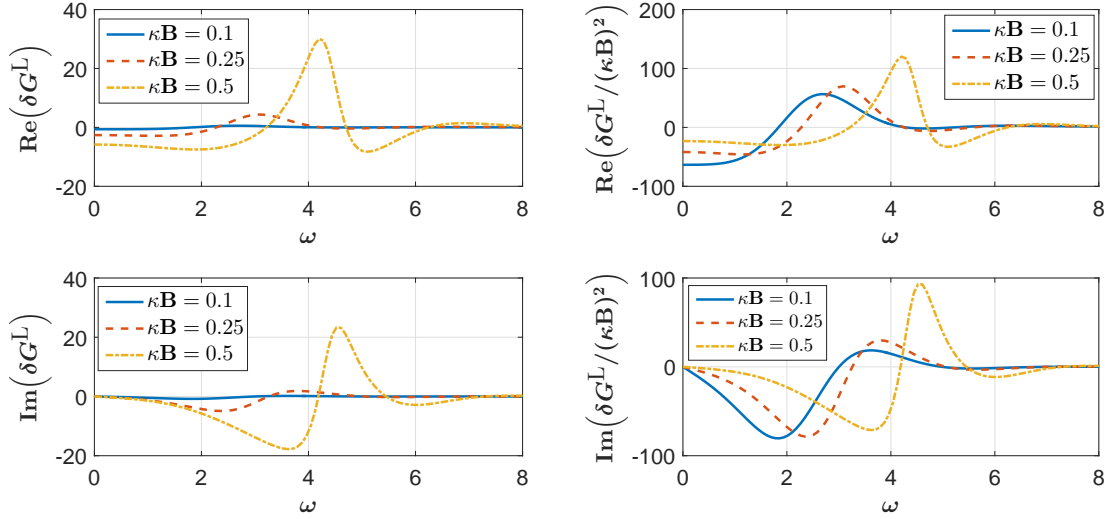


Figure 5. Anomalous correction to correlator: δG^L (left) and $\delta G^L / (\kappa \mathbf{B})^2$ (right).

5 Conclusions

In this paper we continued explorations of the chiral anomaly induced transport within a holographic model containing two $U(1)$ fields interacting via Chern-Simons terms. For a finite temperature system, we computed off-shell constitutive relations for the vector/axial currents responding to external electromagnetic fields.

When a static spatially-inhomogeneous magnetic field is the only external field that is turned on, we showed that both the CME and CSE get corrected by derivative terms, see (1.3,1.4). Within the derivative expansion, we analytically calculated corrections up to

third order in the expansion, see (1.6,1.7,3.22,3.23). Apart from the derivative corrections to CME and CSE, the diffusion constant \mathcal{D}_0 was found to receive a negative anomaly-induced correction, see (1.9). The dispersion relation of the chiral magnetic wave was also found to be modified.

In the second part of our study, we focused on the case of time-varying electric and constant magnetic fields without any externally enforced axial charge asymmetry, though the $\vec{E}(t) \cdot \vec{B}$ term in the continuity equation generates the axial charge density ρ_5 (and thus μ_5) dynamically. For such configuration of external fields, we first analysed the most general constitutive relations for the vector/axial currents, see (1.11,1.12). Then, within the derivative expansion, we explicitly calculated the currents up to third order at nonlinear level, see (1.13,1.14). When put on-shell, the axial current \vec{J}_5 is fully nonlinear in the external electric field.

Employing another approximation, we linearised the constitutive relations assuming the electric field to be weak (1.17). Within this approximation the axial current is zero, while the “off-shell” vector current is parameterised by three frequency-dependent transport coefficient functions: the electric conductivity σ_e , and two chiral anomaly-induced conductivities τ_1, τ_2 , see (1.18). In the DC limit, we analytically computed these conductivities, see (1.22,1.23). Then, for generic ω , the numerical plots were presented in section 4.2. Based on these studies, we notice that the anomaly-induced effects get enhanced at some finite frequency ω , whereas the position of the maximum and strength of the effect depends on the external magnetic field. It might be an effect worth looking for experimentally.

Acknowledgements

We would like to thank Dmitri E. Kharzeev, Alex Kovner, Andrey Sadofyev, Derek Teaney, and Ho-Ung Yee for useful discussions related to this work. YB would like to thank KITPC (Beijing) for financial support and hospitality, Physics Department of the University of Connecticut for hospitality where part of this work was done. This work was supported by the ISRAELI SCIENCE FOUNDATION grant #87277111, BSF grant #012124, the People Program (Marie Curie Actions) of the European Union’s Seventh Framework under REA grant agreement #318921; and the Council for Higher Education of Israel under the PBC Program of Fellowships for Outstanding Post-doctoral Researchers from China and India (2015-2016).

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